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HALPHEN PENCILS ON QUARTIC THREEFOLDS

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Abstract. For any smooth quartic threefold in $\mathbb{P}^4$ we classify pencils on it whose general element is an irreducible surface birational to a surface of Kodaira dimension zero.

1. Introduction

Let X be a smooth quartic threefold in $\mathbb{P}^4$. The following result is proved in [4].

Theorem 1.1. The threefold X does not contain pencils whose general element is an irreducible surface that is birational to a smooth surface of Kodaira dimension $-\infty$.

On the other hand, one can easily see that the threefold X contains infinitely many pencils whose general elements are irreducible surfaces of Kodaira dimension zero.

Definition 1.2. A Halphen pencil is a one-dimensional linear system whose general element is an irreducible subvariety birational to a smooth variety of Kodaira dimension zero.

The following result is proved in [2].

Theorem 1.3. Suppose that X is general. Then every Halphen pencil on X is cut out by

$$\lambda l_1\big(x, y, z, t, w\big) + \mu l_2\big(x, y, z, t, w\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big) \cong \mathbb{P}^4,$$

where l_1 and l_2 are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$.

The assertion of Theorem 1.3 is erroneously proved in [1] without the assumption that the threefold X is general. On the other hand, the following example is constructed in [3].

Example 1.4. Suppose that X is given by the equation

$$w^3x + w^2q_2\big(x, y, z, t\big) + wxp_2\big(x, y, z, t\big) + q_4\big(x, y, z, t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big) \cong \mathbb{P}^4,$$

where q_i and p_i are forms of degree i. Let $\mathcal{P}$ be the pencil on X that is cut out by

$$\lambda x^2 + \mu\Big(wx + q_2\big(x, y, z, t\big)\Big) = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then $\mathcal{P}$ is a Halphen pencil if $q_2(0, y, z, t) \neq 0$ by [2, Theorem 2.3].

The purpose of this paper is to prove the following result.

Theorem 1.5. Let $\mathcal{M}$ be a Halphen pencil on X. Then

- either $\mathcal{M}$ is cut out on X by the pencil

$$\lambda l_1\big(x, y, z, t, w\big) + \mu l_2\big(x, y, z, t, w\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big) \cong \mathbb{P}^4,$$

  where l_1 and l_2 are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$,

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We assume that all varieties are projective, normal and defined over $\mathbb{C}$.

- or the threefold X can be given by the equation

$$w^3x+w^2q_2\big(x,y,z,t\big)+wxp_2\big(x,y,z,t\big)+q_4\big(x,y,z,t\big)=0\subset\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)\cong\mathbb{P}^4$$

  such that $q_2(0,y,z,t)\neq 0$, and $\mathcal{M}$ is cut out on the threefold X by the pencil

$$\lambda x^2+\mu\Big(wx+q_2\big(x,y,z,t\big)\Big)=0,$$

  where q_i and p_i are forms of degree i, and $(\lambda:\mu)\in\mathbb{P}^1$.

Let P be an arbitrary point of the quartic hypersurface $X\subset\mathbb{P}^4$.

Definition 1.6. The mobility threshold of the threefold X at the point P is the number

$$\iota\big(P\big)=\sup\Big\{\lambda\in\mathbb{Q}\ \text{such that}\ \Big|n\Big(\pi^*\big(-K_X\big)-\lambda E\Big)\Big|\ \text{has no fixed components for}\ n\gg 0\Big\},$$

where $\pi\colon Y\to X$ is the ordinary blow up of P, and E is the exceptional divisor of π.

Arguing as in the proof of Theorem 1.5, we obtain the following result.

Theorem 1.7. The following conditions are equivalent:

- the equality $\iota(P)=2$ holds,
- the threefold X can be given by the equation

$$w^3x+w^2q_2\big(x,y,z,t\big)+wxp_2\big(x,y,z,t\big)+q_4\big(x,y,z,t\big)=0\subset\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)\cong\mathbb{P}^4,$$

  where q_i and p_i are forms of degree i such that

$$q_2\big(0,y,z,t\big)\neq 0,$$

  and P is given by the equations $x=y=z=t=0$.

One can easily check that $2\geqslant\iota(P)\geqslant 1$. Similarly, one can show that

- $\iota(P)=1\iff$ the hyperplane section of X that is singular at P is a cone,
- $\iota(P)=3/2\iff$ the threefold X contains no lines passing through P.

The proof of Theorem 1.5 is completed on board of IL-96-300 *Valery Chkalov* while flying from Seoul to Moscow. We thank Aeroflot Russian Airlines for good working conditions.

2. Important lemma

Let S be a surface, let O be a smooth point of S, let R be an effective Weil divisor on the surface S, and let $\mathcal{D}$ be a linear system on the surface S that has no fixed components.

Lemma 2.1. Let D_1 and D_2 be general curves in $\mathcal{D}$. Then

$$\mathrm{mult}_O\Big(D_1\cdot R\Big)=\mathrm{mult}_O\Big(D_2\cdot R\Big)\leqslant\mathrm{mult}_O\big(R\big)\mathrm{mult}_O\Big(D_1\cdot D_1\Big).$$

Proof. Put $S_0=S$ and $O_0=O$. Let us consider the sequence of blow ups

$$S_n\xrightarrow{\ \pi_n\ }S_{n-1}\xrightarrow{\ \pi_{n-1}\ }\cdots\xrightarrow{\ \pi_2\ }S_1\xrightarrow{\ \pi_1\ }S_0$$

such that π_1 is a blow up of the point O_0, and π_i is a blow up of the point O_{i-1} that is contained in the curve E_{i-1}, where E_{i-1} is the exceptional curve of π_{i-1}, and $i=2,\ldots,n$.

Let D_j^i be the proper transform of D_j on S_i for $i=0,\ldots,n$ and $j=1,2$. Then

$$D_1^i\equiv D_2^i\equiv\pi_i^*\Big(D_1^{i-1}\Big)-\mathrm{mult}_{O_{i-1}}\Big(D_1^{i-1}\Big)E_i\equiv\pi_i^*\Big(D_2^{i-1}\Big)-\mathrm{mult}_{O_{i-1}}\Big(D_2^{i-1}\Big)E_i$$

for $i=1,\ldots,n$. Put $d_i=\mathrm{mult}_{O_{i-1}}(D_1^{i-1})=\mathrm{mult}_{O_{i-1}}(D_2^{i-1})$ for $i=1,\ldots,n$.

Let R^i be the proper transform of R on the surface S_i for $i = 0, \ldots, n$. Then

$$R^i \equiv \pi_i^*\Big(R^{i-1}\Big) - \mathrm{mult}_{O_{i-1}}\Big(R^{i-1}\Big)E_i$$

for $i = 1, \ldots, n$. Put $r_i = \mathrm{mult}_{O_{i-1}}(R^{i-1})$ for $i = 1, \ldots, n$. Then $r_1 = \mathrm{mult}_O(R)$.

We may chose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D_1^n \cap D_2^n$ is empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then

$$\mathrm{mult}_O\Big(D_1 \cdot D_2\Big) = \sum_{i=1}^{n} d_i^2.$$

We may chose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D_1^n \cap R^n$ and $D_2^n \cap R^n$ are empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then

$$\mathrm{mult}_O\Big(D_1 \cdot R\Big) = \mathrm{mult}_O\Big(D_2 \cdot R\Big) = \sum_{i=1}^{n} d_i r_i,$$

where some numbers among $r_1, \ldots, r_n$ may be zero. Then

$$\mathrm{mult}_O\Big(D_1 \cdot R\Big) = \mathrm{mult}_O\Big(D_2 \cdot R\Big) = \sum_{i=1}^{n} d_i r_i \leqslant \sum_{i=1}^{n} d_i r_1 \leqslant \sum_{i=1}^{n} d_i^2 r_1 = \mathrm{mult}_O\big(R\big)\mathrm{mult}_O\Big(D_1 \cdot D_2\Big),$$

because $d_i \leqslant d_i^2$ and $r_i \leqslant r_1 = \mathrm{mult}_O(R)$ for every $i = 1, \ldots, n$. □

The assertion of Lemma 2.1 is a cornerstone of the proof of Theorem 1.5.

3. Curves

Let X be a smooth quartic threefold in $\mathbb{P}^4$, let $\mathcal{M}$ be a Halphen pencil on X. Then

$$\mathcal{M} \sim -nK_X,$$

since $\mathrm{Pic}(X) = \mathbb{Z}K_X$. Put $\mu = 1/n$. Then

- the log pair $(X, \mu\mathcal{M})$ is canonical by [3, Theorem A],
- the log pair $(X, \mu\mathcal{M})$ is not terminal by [2, Theorem 2.1].

Let $\mathbb{CS}(X, \mu\mathcal{M})$ be the set of non-terminal centers of $(X, \mu\mathcal{M})$ (see [2]). Then

$$\mathbb{CS}\Big(X, \mu\mathcal{M}\Big) \neq \varnothing,$$

because $(X, \mu\mathcal{M})$ is not terminal. Let M_1 and M_2 be two general surfaces in $\mathcal{M}$.

Lemma 3.1. Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then

$$\mathrm{mult}_P\big(M\big) = n\mathrm{mult}_P\big(T\big) = 2n,$$

where M is any surface in $\mathcal{M}$, and T is the surface in $|-K_X|$ that is singular at P.

Proof. It follows from [6, Proposition 1] that the inequality

$$\mathrm{mult}_P\Big(M_1 \cdot M_2\Big) \geqslant 4n^2$$

holds. Let H be a general surface in $|-K_X|$ such that $P \in H$. Then

$$4n^2 = H \cdot M_1 \cdot M_2 \geqslant \mathrm{mult}_P\Big(M_1 \cdot M_2\Big) \geqslant 4n^2,$$

which gives $(M_1 \cdot M_2)_P = 4n^2$. Arguing as in the proof of [6, Proposition 1], we see that

$$\mathrm{mult}_P\big(M_1\big) = \mathrm{mult}_P\big(M_2\big) = 2n,$$

because $(M_1 \cdot M_2)_P = 4n^2$. Similarly, we see that

$$4n = H \cdot T \cdot M_1 \geqslant \mathrm{mult}_P\big(T\big)\mathrm{mult}_P\big(M_1\big) = 2n\mathrm{mult}_P\big(T\big) \geqslant 4n,$$

which implies that $\mathrm{mult}_P(T) = 2$. Finally, we also have

$$4n^2 = H \cdot M \cdot M_1 \geqslant \mathrm{mult}_P\big(M\big)\mathrm{mult}_P\big(M_1\big) = 2n\mathrm{mult}_P\big(M\big) \geqslant 4n^2,$$

where M is any surface in $\mathcal{M}$, which completes the proof. □

Lemma 3.2. Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then

$$M_1 \cap M_2 = \bigcup_{i=1}^{r} L_i,$$

where $L_1, \ldots, L_r$ are lines on the threefold X that pass through the point P.

Proof. Let H be a general surface in $|-K_X|$ such that $P \in H$. Then

$$4n^2 = H \cdot M_1 \cdot M_2 = \mathrm{mult}_P\Big(M_1 \cdot M_2\Big) = 4n^2$$

by Lemma 3.1. Then $\mathrm{Supp}(M_1 \cdot M_2)$ consists of lines on X that pass through P. □

Lemma 3.3. Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then

$$n/3 \leqslant \mathrm{mult}_L\big(\mathcal{M}\big) \leqslant n/2$$

for every line $L \subset X$ that passes through the point P.

Proof. Let D be a general hyperplane section of X through L. Then we have

$$M\Big|_D = \mathrm{mult}_L\big(\mathcal{M}\big)L + \Delta,$$

where M is a general surface in $\mathcal{M}$ and Δ is an effective divisor such that

$$\mathrm{mult}_P\big(\Delta\big) \geqslant 2n - \mathrm{mult}_L\big(\mathcal{M}\big).$$

On the surface D we have $L \cdot L = -2$. Then

$$n = \Big(\mathrm{mult}_L\big(\mathcal{M}\big)L + \Delta\Big) \cdot L = -2\mathrm{mult}_L\big(\mathcal{M}\big) + \Delta \cdot L$$

on the surface D. But $\Delta \cdot L \geqslant \mathrm{mult}_P(\Delta) \geqslant 2n - \mathrm{mult}_L(\mathcal{M})$. Thus, we get

$$n \geqslant -2\mathrm{mult}_L\big(\mathcal{M}\big) + \mathrm{mult}_P\big(\Delta\big) \geqslant 2n - 3\mathrm{mult}_L\big(\mathcal{M}\big),$$

which implies that $n/3 \leqslant \mathrm{mult}_L\big(\mathcal{M}\big)$.

Let T be the surface in $|-K_X|$ that is singular at P. Then $T \cdot D$ is reduced and

$$T \cdot D = L + Z,$$

where Z is an irreducible plane cubic curve such that $P \in Z$. Then

$$3n = \Big(\mathrm{mult}_L\big(\mathcal{M}\big)L + \Delta\Big) \cdot Z = 3\mathrm{mult}_L\big(\mathcal{M}\big) + \Delta \cdot Z$$

on the surface D. The set $\Delta \cap Z$ is finite by Lemma 3.2. In particular, we have

$$\Delta \cdot Z \geqslant \mathrm{mult}_P\big(\Delta\big) \geqslant 2n - \mathrm{mult}_L\big(\mathcal{M}\big),$$

because $\mathrm{Supp}(\Delta)$ does not contain the curve Z. Thus, we get

$$3n \geqslant 3\mathrm{mult}_L\big(\mathcal{M}\big) + \mathrm{mult}_P\big(\Delta\big) \geqslant 2n + 2\mathrm{mult}_L\big(\mathcal{M}\big),$$

which implies that $\mathrm{mult}_L\big(\mathcal{M}\big) \leqslant n/2$. □

In the rest of this section we prove the following result.

Proposition 3.4. Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a curve. Then $n = 1$.

Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a curve Z. Then it follows Lemmas 3.2 and 3.3 that the set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain points of the threefold X and

$$\mathrm{mult}_Z\big(\mathcal{M}\big) = n, \tag{3.5}$$

because $(X, \mu\mathcal{M})$ is canonical but not terminal. Then $\deg(Z) \leqslant 4$ by [2, Lemma 2.1].

Lemma 3.6. Suppose that $\deg(Z) = 1$. Then $n = 1$.

Proof. Let $\pi : V \to X$ be the blow up of X along the line Z. Let $\mathcal{B}$ be the proper transform of the pencil $\mathcal{M}$ on the threefold V, and let B be a general surface in $\mathcal{B}$. Then

$$B \sim -nK_V \tag{3.7}$$

by (3.5). There is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the projection from the line Z and η is the morphism induced by the linear system $|-K_V|$. Thus, it follows from (3.7) that $\mathcal{B}$ is the pull-back of a pencil $\mathcal{P}$ on $\mathbb{P}^2$ by η.

We see that the base locus of $\mathcal{B}$ is contained in the union of fibers of η.

The set $\mathbb{CS}(V, \mu\mathcal{B})$ is not empty by [2, Theorem 2.1]. It easily follows from (3.5) that the set $\mathbb{CS}(V, \mu\mathcal{B})$ does not contain points because $\mathbb{CS}(X, \mu\mathcal{M})$ contains no points.

We see that there is an irreducible curve $L \subset V$ such that

$$\mathrm{mult}_L\big(\mathcal{B}\big) = n$$

and $\eta(L)$ is a point $Q \in \mathbb{P}^2$. Let C be a general curve in $\mathcal{P}$. Then $\mathrm{mult}_Q(C) = n$. But

$$C \sim \mathcal{O}_{\mathbb{P}^2}\big(n\big)$$

by (3.7). Thus, we see that $n = 1$, because general surface in $\mathcal{M}$ is irreducible. $\square$

Thus, we may assume that the set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain lines.

Lemma 3.8. The curve $Z \subset \mathbb{P}^4$ is contained in a plane.

Proof. Suppose that Z is not contained in any plane in $\mathbb{P}^4$. Let us show that this assumption leads to a contradiction. Since $\deg(Z) \leqslant 4$, we have

$$\deg\big(Z\big) \in \big\{3, 4\big\},$$

and Z is smooth if $\deg(Z) = 3$. If $\deg(Z) = 4$, then Z may have at most one double point.

Suppose that Z is smooth. Let $\alpha\colon U \to X$ be the blow up at Z, and let F be the exceptional divisor of the morphism α. Then the base locus of the linear system

$$\Big|\alpha^*\Big(-\deg\big(Z\big)K_X\Big) - F\Big|$$

does not contain any curve. Let D_1 and D_2 be the proper transforms on U of two sufficiently general surfaces in the linear system $\mathcal{M}$. Then it follows from (3.5) that

$$\left(\alpha^*\Big(-\deg\big(Z\big)K_X\Big)-F\right)\cdot D_1\cdot D_2 = n^2\left(\alpha^*\Big(-\deg\big(Z\big)K_X\Big)-F\right)\cdot\left(\alpha^*\Big(-K_X\Big)-F\right)^2 \geqslant 0,$$

because the cycle $D_1 \cdot D_2$ is effective. On the other hand, we have

$$\Big(\alpha^*\Big(-\deg\big(Z\big)K_X\Big)-F\Big)\cdot\Big(\alpha^*\Big(-K_X\Big)-F\Big)^2=\Bigg(3\deg\big(Z\big)-\Big(\deg\big(Z\big)\Big)^2-2\Bigg)<0,$$

which is a contradiction. Thus, the curve Z is not smooth.

Thus, we see that Z is a quartic curve with a double point O.

Let $\beta\colon W\to X$ be the composition of the blow up of the point O with the blow up of the proper transform of the curve Z. Let G and E be the exceptional surfaces of the morphism β such that $\beta(E)=Z$ and $\beta(G)=O$. Then the base locus of the linear system

$$\Big|\beta^*\big(-4K_X\big)-E-2G\Big|$$

does not contain any curve. Let R_1 and R_2 be the proper transforms on W of two sufficiently general surfaces in $\mathcal{M}$. Put $m=\mathrm{mult}_O(\mathcal{M})$. Then it follows from (3.5) that

$$\Bigg(\beta^*\Big(-4K_X\Big)-E-2G\Bigg)\cdot R_1\cdot R_2=\Bigg(\beta^*\Big(-4K_X\Big)-E-2G\Bigg)\cdot\Bigg(\beta^*\Big(-nK_X\Big)-nE-mG\Bigg)^2\geqslant 0,$$

and $m<2n$, because the set $\mathbb{CS}(X,\mu\mathcal{M})$ does not contain points. Then

$$\Bigg(\beta^*\Big(-4K_X\Big)-E-2G\Bigg)\cdot\Bigg(\beta^*\Big(-nK_X\Big)-nE-mG\Bigg)^2=-8n^2+6mn-m^2<0,$$

which is a contradiction. □

If $\deg(Z)=4$, then $n=1$ by Lemma 3.8 and [2, Theorem 2.2].

Lemma 3.9. Suppose that $\deg(Z)=3$. Then $n=1$.

Proof. Let $\mathcal{P}$ be the pencil in $|-K_X|$ that contains all hyperplane sections of X that pass through the curve Z. Then the base locus of $\mathcal{P}$ consists of the curve Z and a line $L\subset X$.

Let D be a sufficiently general surface in the pencil $\mathcal{P}$, and let M be a sufficiently general surface in the pencil $\mathcal{M}$. Then D is a smooth surface, and

$$M\Big|_D=nZ+\mathrm{mult}_L\big(\mathcal{M}\big)L+B\equiv nZ+nL, \tag{3.10}$$

where B is a curve whose support does not contain neither Z nor L.

On the surface D, we have $Z\cdot L=3$ and $L\cdot L=-2$. Intersecting (3.10) with L, we get

$$n=\big(nZ+nL\big)\cdot L=3n-2\mathrm{mult}_L\big(\mathcal{M}\big)+B\cdot L\geqslant 3n-2\mathrm{mult}_L\big(\mathcal{M}\big),$$

which easily implies that $\mathrm{mult}_L(\mathcal{M})\geqslant n$. But the inequality $\mathrm{mult}_L(\mathcal{M})\geqslant n$ is impossible, because we assumed that $\mathbb{CS}(X,\mu\mathcal{M})$ contains no lines. □

Lemma 3.11. Suppose that $\deg(Z)=2$. Then $n=1$.

Proof. Let $\alpha\colon U\to X$ be the blow up of the curve Z. Then $|-K_U|$ is a pencil, whose base locus consists of a smooth irreducible curve $L\subset U$.

Let D be a general surface in $|-K_U|$. Then D is a smooth surface.

Let $\mathcal{B}$ be the proper transform of the pencil $\mathcal{M}$ on the threefold U. Then

$$-nK_U\Big|_D\equiv B\Big|_D\equiv nL,$$

where B is a general surface in $\mathcal{B}$. But $L^2 = -2$ on the surface D. Then

$$L \in \mathbb{CS}\Big(U, \mu\mathcal{B}\Big)$$

which implies that $\mathcal{B} = \big| -K_U\big|$ by [2, Theorem 2.2]. Then $n = 1$. □

The assertion of Proposition 3.4 is proved.

4. Points

Let X be a smooth quartic threefold in $\mathbb{P}^4$, let $\mathcal{M}$ be a Halphen pencil on X. Then

$$\mathcal{M} \sim -nK_X,$$

since $\mathrm{Pic}(X) = \mathbb{Z}K_X$. Put $\mu = 1/n$. Then

- the log pair $(X, \mu\mathcal{M})$ is canonical by [3, Theorem A],
- the log pair $(X, \mu\mathcal{M})$ is not terminal by [2, Theorem 2.1].

Remark 4.1. To prove Theorem 1.5, it is enough to show that X can be given by

$$w^3x + w^2q_2\big(x, y, z, t\big) + wxp_2\big(x, y, z, t\big) + q_4\big(x, y, z, t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big) \cong \mathbb{P}^4,$$

where q_i and p_i are homogeneous polynomials of degree $i \geqslant 2$ such that $q_2(0, y, z, t) \neq 0$.

Let $\mathbb{CS}(X, \mu\mathcal{M})$ be the set of non-terminal centers of $(X, \mu\mathcal{M})$ (see [2]). Then

$$\mathbb{CS}\Big(X, \mu\mathcal{M}\Big) \neq \varnothing,$$

because $(X, \mu\mathcal{M})$ is not terminal. Suppose that $n \neq 1$. There is a point $P \in X$ such that

$$P \in \mathbb{CS}\Big(X, \mu\mathcal{M}\Big)$$

by Proposition 3.4. It follows from Lemmas 3.1, 3.2 and 3.3 that

- there are finitely many distinct lines $L_1, \ldots, L_r \subset X$ containing $P \in X$,
- the equality $\mathrm{mult}_P(M) = 2n$ holds, and

  $$n/3 \leqslant \mathrm{mult}_{L_i}\big(M\big) \leqslant n/2,$$

  where M is a general surface in the pencil $\mathcal{M}$,
- the equality $\mathrm{mult}_P(T) = 2$ holds, where $T \in |-K_X|$ such that $\mathrm{mult}_P(T) \geqslant 2$,
- the base locus of the pencil $\mathcal{M}$ consists of the lines $L_1, \ldots, L_r$, and

  $$\mathrm{mult}_P\Big(M_1 \cdot M_2\Big) = 4n^2,$$

  where M_1 and M_2 are sufficiently general surfaces in $\mathcal{M}$.

Lemma 4.2. The equality $\mathbb{CS}(X, \mu\mathcal{M}) = \{P\}$ holds.

Proof. The set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain curves by Proposition 3.4.

Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $Q \in X$ such that $Q \neq P$. Then $r = 1$.

Let D be a general hyperplane section of X that passes through L_1. Then

$$M\Big|_D = \mathrm{mult}_{L_1}\big(\mathcal{M}\big)L_1 + \Delta,$$

where M is a general surface in $\mathcal{M}$ and Δ is an effective divisor such that

$$\mathrm{mult}_P\big(\Delta\big) \geqslant 2n - \mathrm{mult}_{L_1}\big(\mathcal{M}\big) \leqslant \mathrm{mult}_Q\big(\Delta\big).$$

On the surface D, we have $L_1^2 = -2$. Then

$$n = \Big(\mathrm{mult}_{L_1}\big(\mathcal{M}\big)L_1 + \Delta\Big)\cdot L_1 = -2\mathrm{mult}_{L_1}\big(\mathcal{M}\big) + \Delta\cdot L \geqslant -2\mathrm{mult}_{L_1}\big(\mathcal{M}\big) + 2\Big(2n - \mathrm{mult}_{L_1}\big(\mathcal{M}\big)\Big),$$

which gives $\mathrm{mult}_{L_1}(\mathcal{M}) \geqslant 3n/4$. But $\mathrm{mult}_{L_1}(\mathcal{M}) \leqslant n/2$ by Lemma 3.3. $\square$

The quartic threefold X can be given by an equation

$$w^3x + w^2q_2\big(x,y,z,t\big) + wq_3\big(x,y,z,t\big) + q_4\big(x,y,z,t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big) \cong \mathbb{P}^4,$$

where q_i is a homogeneous polynomial of degree $i \geqslant 2$.

Remark 4.3. The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = q_2\big(x,y,z,t\big) = q_3\big(x,y,z,t\big) = q_4\big(x,y,z,t\big) = 0,$$

the surface T is cut out on X by $x = 0$, and $\mathrm{mult}_P(T) = 2 \iff q_2(0,y,z,t) \neq 0$.

Let $\pi\colon V \to X$ be the blow up of the point P, let E be the π-exceptional divisor. Then

$$\mathcal{B} \equiv \pi^*\big(-nK_X\big) - 2nE \equiv -nK_V,$$

where $\mathcal{B}$ is the proper transform of the pencil $\mathcal{M}$ on the threefold V.

Remark 4.4. The pencil $\mathcal{B}$ has no base curves in E, because

$$\mathrm{mult}_P\big(M_1\cdot M_2\big) = \mathrm{mult}_P\big(M_1\big)\mathrm{mult}_P\big(M_2\big).$$

Let $\bar{L}_i$ be the proper transform of the line L_i on the threefold V for $i = 1, \ldots, r$. Then

$$B_1\cdot B_2 = \sum_{i=1}^{r}\mathrm{mult}_{\bar{L}_i}\Big(B_1\cdot B_2\Big)\bar{L}_i,$$

where B_1 and B_2 are proper transforms of M_1 and M_2 on the threefold V, respectively.

Lemma 4.5. Let Z be an irreducible curve on X such that $Z \notin \{L_1, \ldots, Z_r\}$. Then

$$\deg\big(Z\big) \geqslant 2\mathrm{mult}_P\big(Z\big),$$

and the equality $\deg(Z) = 2\mathrm{mult}_P(Z)$ implies that

$$\bar{Z} \cap \left(\bigcup_{i=1}^{r}\bar{L}_i\right) = \varnothing,$$

where $\bar{Z}$ is a proper transform of the curve Z on the threefold V.

Proof. The curve $\bar{Z}$ is not contained in the base locus of the pencil $\mathcal{B}$. Then

$$0 \leqslant B_i\cdot\bar{Z} \leqslant n\Big(\deg\big(Z\big) - 2\mathrm{mult}_P\big(Z\big)\Big),$$

which implies the required assertions. $\square$

To conclude the proof of Theorem 1.5, it is enough to show that

$$q_3\big(x,y,z,t\big) = xp_2\big(x,y,z,t\big) + q_2\big(x,y,z,t\big)p_1\big(x,y,z,t\big),$$

where p_1 and p_2 are some homogeneous polynomials of degree 1 and 2, respectively.

5. Good points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0,y,z,t)=0\subset \mathrm{Proj}\Big(\mathbb{C}[y,z,t]\Big)\cong\mathbb{P}^2$$

is reduced and irreducible. In this section we prove the following result.

Proposition 5.1. The polynomial $q_3(0,y,z,t)$ is divisible by $q_2(0,y,z,t)$.

Let us prove Proposition 5.1. Suppose that $q_3(0,y,z,t)$ is not divisible by $q_2(0,y,z,t)$.
Let $\mathcal{R}$ be the linear system on the threefold X that is cut out by quadrics

$$xh_1(x,y,z,t)+\lambda\Big(wx+q_2(x,y,z,t)\Big)=0,$$

where h_1 is an arbitrary linear form and $\lambda\in\mathbb{C}$. Then $\mathcal{R}$ does not have fixed components.

Lemma 5.2. Let R_1 and R_2 be general surfaces in the linear system $\mathcal{R}$. Then

$$\sum_{i=1}^{r}\mathrm{mult}_{L_i}\Big(R_1\cdot R_2\Big)\leqslant 6.$$

Proof. We may assume that R_1 is cut out by the equation

$$wx+q_2(x,y,z,t)=0,$$

and R_2 is cut out by $xh_1(x,y,z,t)=0$, where h_1 is sufficiently general. Then

$$\mathrm{mult}_{L_i}\Big(R_1\cdot R_2\Big)=\mathrm{mult}_{L_i}\Big(R_1\cdot T\Big).$$

Put $m_i=\mathrm{mult}_{L_i}(R_1\cdot T)$. Then

$$R_1\cdot T=\sum_{i=1}^{r}m_iL_i+\Delta,$$

where $m_i\in\mathbb{N}$, and Δ is a cycle, whose support contains no lines passing through P.
Let $\bar{R}_1$ and $\bar{T}$ be the proper transforms of R_1 and T on V, respectively. Then

$$\bar{R}_1\cdot\bar{T}=\sum_{i=1}^{r}m_i\bar{L}_i+\Omega,$$

where Ω is an effective cycle, whose support contains no lines passing through P.
The support of the cycle Ω does not contain curves that are contained in the exceptional divisor E, because $q_3(0,y,z,t)$ is not divisible by $q_2(0,y,z,t)$ by our assumption. Then

$$6=E\cdot\bar{R}_1\cdot\bar{T}=\sum_{i=1}^{r}m_i\big(E\cdot\bar{L}_i\big)+E\cdot\Omega\geqslant\sum_{i=1}^{r}m_i\big(E\cdot\bar{L}_i\big)=\sum_{i=1}^{r}m_i,$$

which is exactly what we want. □

Let M and R be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M\cdot R=\sum_{i=1}^{r}m_iL_i+\Delta,$$

where $m_i\in\mathbb{N}$, and Δ is a cycle, whose support contains no lines passing through P.

Lemma 5.3. The cycle Δ is not trivial.

Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil. $\square$

We have $\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i$. On the other hand, the inequality

$$\mathrm{mult}_P\big(\Delta\big) \geqslant 6n - \sum_{i=1}^{r} m_i$$

holds, because $\mathrm{mult}_P(M) = 2n$ and $\mathrm{mult}_P(R) \geqslant 3$. It follows from Lemma 4.5 that

$$\deg\big(\Delta\big) = 8n - \sum_{i=1}^{r} m_i \geqslant 2\mathrm{mult}_P\big(\Delta\big) \geqslant 2\left(6n - \sum_{i=1}^{r} m_i\right),$$

which implies that $\sum_{i=1}^{r} m_i \geqslant 4n$. But it follows from Lemmas 2.1 and 3.3 that

$$m_i \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)\mathrm{mult}_{L_i}\big(M\big) \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2$$

for every $i = 1, \ldots, r$, where R_1 and R_2 are general surfaces in $\mathcal{R}$. Then

$$\sum_{i=1}^{r} m_i \leqslant \sum_{i=1}^{r} \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2 \leqslant 3n$$

by Lemma 5.2, which is a contradiction.

The assertion of Proposition 5.1 is proved.

6. Bad points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[y, z, t]\Big) \cong \mathbb{P}^2$$

is reduced and reducible. Therefore, we have

$$q_2(x, y, z, t) = \big(\alpha_1 y + \beta_1 z + \gamma_1 t\big)\big(\alpha_2 y + \beta_2 z + \gamma_2 t\big) + xp_1\big(x, y, z, t\big)$$

where $p_1(x, y, z, t)$ is a linear form, and $(\alpha_1 : \beta_1 : \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2 : \beta_2 : \gamma_2)$.

Proposition 6.1. The polynomial $q_3(0, y, z, t)$ is divisible by $q_2(0, y, z, t)$.

Suppose that $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$. Then without loss of generality, we may assume that $q_3(0, y, z, t)$ is not divisible by $\alpha_1 y + \beta_1 z + \gamma_1 t$.

Let Z be the curve in X that is cut out by the equations

$$x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0.$$

Remark 6.2. The equality $\mathrm{mult}_P(Z) = 3$ holds, but Z is not necessary reduced.

Hence, it follows from Lemma 4.5 that $\mathrm{Supp}(Z)$ contains a line among $L_1, \ldots, L_r$.

Lemma 6.3. The support of the curve Z does not contain an irreducible conic.

Proof. Suppose that $\mathrm{Supp}(Z)$ contains an irreducible conic C. Then

$$Z = C + L_i + L_j$$

for some $i \in \{1, \ldots, r\} \ni j$. Then $i = j$, because otherwise the set

$$\Big(C \cap L_i\Big) \bigcup \Big(C \cap L_j\Big)$$

contains a point that is different from P, which is impossible by Lemma 4.5. We see that

$$Z = C + 2L_i,$$

and it follows from Lemma 4.5 that $C \cap L_i = P$. Then C is tangent to L_i at the point P
Let $\bar{C}$ be a proper transform of the curve C on the threefold V. Then

$$\bar{C} \cap \bar{L}_i \neq \varnothing,$$

which is impossible by Lemma 4.5. The assertion is proved. ☐

Lemma 6.4. The support of the curve Z consists of lines.

Proof. Suppose that $\mathrm{Supp}(Z)$ does not consist of lines. It follows from Lemma 6.3 that

$$Z = L_i + C,$$

where C is an irreducible cubic curve. But $\mathrm{mult}_P(Z) = 3$. Then

$$\mathrm{mult}_P\big(C\big) = 2,$$

which is impossible by Lemma 4.5 ☐

We may assume that there is a line $L \subset X$ such that $P \not\in P$ and

$$Z = a_1 L_1 + \cdots + a_k L_k + L,$$

where $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 \geqslant a_2 \geqslant a_3$ and $\sum_{i=1}^{k} a_i = 3$.

Remark 6.5. We have $L_i \neq L_j$ whenever $i \neq j$.

Let H be a sufficiently general surface of X that is cut out by the equation

$$\lambda x + \mu\Big(\alpha_1 y + \beta_1 z + \gamma_1 t\Big) = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then H has at most isolated singularities.

Remark 6.6. The surface H is smooth at the points P and $L \cap L_i$, where $i = 1, \ldots, k$.

Let $\bar{H}$ and $\bar{L}$ be the proper transforms of H and L on the threefold V, respectively.

Lemma 6.7. The inequality $k \neq 3$ holds.

Proof. Suppose that the equality $k = 3$ holds. Then H is smooth. Put

$$B\Big|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is an effective divisor on $\bar{H}$ whose support does not contain any of the curves $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$. Then

$$\bar{L} \not\subseteq \mathrm{Supp}\big(\Omega\big) \not\supseteq \bar{H} \cap E,$$

because the base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = \bar{L} \cdot \Big(m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega\Big) = \sum_{i=1}^{3} m_i + \bar{L} \cdot \Omega \geqslant \sum_{i=1}^{3} m_i,$$

which implies that $\sum_{i=1}^{3} m_i \leqslant n$. On the other hand, we have

$$-n = \bar{L}_i \cdot \Big(m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega\Big) = -3m_i + L_i \cdot \Omega \geqslant -3m_i,$$

which implies that $m_i \geqslant n/3$. Thus, we have $m_1 = m_2 = m_3 = n/3$ and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,$$

which implies that $\mathrm{Supp}(\Omega) \cap \bar{L}_1 = \mathrm{Supp}(\Omega) \cap \bar{L}_2 = \mathrm{Supp}(\Omega) \cap \bar{L}_3 = \varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}} = \frac{n}{3}\Big(\bar{L}_1 + \bar{L}_2 + \bar{L}_3\Big) + \Omega',$$

where Ω' is an effective divisor on the surface $\bar{H}$ such that

$$\mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_1 = \mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_2 = \mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_3 = \varnothing.$$

One can easily check that $\Omega \cdot \Omega' = n^2 \neq 0$. Then

$$\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big) \neq \varnothing,$$

because $|\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega')| < +\infty$ due to generality of the surfaces B and B'.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Hence, we have

$$\mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) = \bigcup_{i=1}^{r} \bar{L}_i,$$

but $\bar{L}_i \cap \bar{H} = \varnothing$ whenever $i \notin \{1, 2, 3\}$. Hence, we have

$$\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup \big(\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big)\big) = \mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) \cap \bar{H} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3,$$

which implies that $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3$. In particular, we see that

$$\mathrm{Supp}(\Omega) \cap \Big(\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3\Big) \neq \varnothing,$$

because $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \neq \varnothing$. But $\mathrm{Supp}(\Omega) \cap \bar{L}_i = \varnothing$ for $i = 1, 2, 3$. $\square$

Lemma 6.8. The inequality $k \neq 2$ holds.

Proof. Suppose that the equality $k = 2$ holds. Then $Z = 2L_1 + L_2 + L$. Put

$$B\Big|_{\bar{H}} = m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is an effective divisor on $\bar{H}$ whose support does not contain the curves $\bar{L}_1$ and $\bar{L}_2$. Then $\bar{L} \not\subseteq \mathrm{Supp}(\Omega) \not\supseteq \bar{H} \cap E$ and

$$n = \bar{L} \cdot \Big(m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega\Big) = m_1 + m_2 + \bar{L} \cdot \Omega \geqslant m_1 + m_2,$$

which implies that $m_1 + m_2 \leqslant n$. On the other hand, we have

$$\bar{T}|_{\bar{H}} = 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E\Big|_{\bar{H}} \equiv \Big(\pi^*\Big(-K_X\Big) - 2E\Big)\Big|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface T on the threefold V. Then

$$-1 = \bar{L}_1 \cdot \Big(2\bar{L}_1 + \bar{L}_2 + \bar{L} + E\Big|_{\bar{H}}\Big) = 2\Big(\bar{L}_1 \cdot \bar{L}_1\Big) + 2,$$

which implies that $\bar{L}_1 \cdot \bar{L}_1 = -3/2$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_1 \cdot \Big(m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega\Big) = -3m_1/2 + L_1 \cdot \Omega \geqslant -3m_1/2,$$

which gives $m_1 \geqslant 2n/3$. Similarly, we see that $\bar{L}_2 \cdot \bar{L}_2 = -3$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_2 \cdot \Big(m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega\Big) = -3m_2 + L_2 \cdot \Omega \geqslant -3m_2,$$

which implies that $m_2 \leqslant n/3$. Thus, we have $m_1 = 2m_2 = 2n/3$ and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = 0,$$

which implies that $\mathrm{Supp}(\Omega) \cap \bar{L}_1 = \mathrm{Supp}(\Omega) \cap \bar{L}_2 = \varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}} = \frac{2n}{3}\bar{L}_1 + \frac{n}{3}\bar{L}_2 + \Omega',$$

where Ω' is an effective divisor on $\bar{H}$ whose support does not contain $\bar{L}_1$ and $\bar{L}_2$ such that

$$\mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_1 = \mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_2 = \varnothing,$$

which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that

$$\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big) \neq \varnothing,$$

and arguing as in the proof of Lemma 6.7 we obtain a contradiction. $\square$

It follows from Lemmas 6.7 and 6.8 that $Z = 3L_1 + L$. Put

$$B\Big|_{\bar{H}} = m_1\bar{L}_1 + \Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is a curve such that $\bar{L}_1 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\bar{L} \not\subseteq \mathrm{Supp}\big(\Omega\big) \not\supseteq \bar{H} \cap E,$$

because the base locus of $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = \bar{L} \cdot \Big(m_1\bar{L}_1 + \Omega\Big) = m_1 + \bar{L} \cdot \Omega \geqslant m_1,$$

which implies that $m_1 \leqslant n$. On the other hand, we have

$$\bar{T}|_{\bar{H}} = 3\bar{L}_1 + \bar{L} + E\Big|_{\bar{H}} \equiv \Big(\pi^*\Big(-K_X\Big) - 2E\Big)\Big|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface T on the threefold V. Then

$$-1 = \bar{L}_1 \cdot \Big(3\bar{L}_1 + \bar{L} + E\Big|_{\bar{H}}\Big) = 3\bar{L}_1 \cdot \bar{L}_1 + 2,$$

which implies that $\bar{L}_1 \cdot \bar{L}_1 = -1$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_1 \cdot \Big(m_1\bar{L}_1 + \Omega\Big) = -m_1 + L_1 \cdot \Omega \geqslant -m_1,$$

which gives $m_1 \geqslant n$. Thus, we have $m_1 = n$ and $\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = 0$. Then $\mathrm{Supp}(\Omega) \cap \bar{L}_1 = \varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}} = n\bar{L}_1 + \Omega',$$

where Ω' is an effective divisor on $\bar{H}$ whose support does not contain $\bar{L}_1$ such that

$$\mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_1 = \varnothing,$$

which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \neq \varnothing$.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, L_r$. Hence, we have

$$\mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) = \bigcup_{i=1}^{r} \bar{L}_i,$$

but $\bar{L}_i \cap \bar{H} = \varnothing$ whenever $\bar{L}_i \neq \bar{L}_1$. Then $\mathrm{Supp}(\Omega) \cap \bar{L}_1 \neq \varnothing$, because

$$\bar{L}_1 \cup \Big(\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big)\Big) = \mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) \cap \bar{H} = \bar{L}_1,$$

which is a contradiction. The assertion of Proposition 6.1 is proved.

7. Very bad points

Let us use the assumptions and notation of Section 4. Suppose that $q_2 = y^2$.

The proof of Proposition 6.1 implies that $q_3(0, y, z, t)$ is divisible by y. Then

$$q_3 = yf_2\big(z,t\big) + xh_2\big(z,t\big) + x^2a_1\big(x,y,z,t\big) + xyb_1\big(x,y,z,t\big) + y^2c_1\big(y,z,t\big)$$

where a_1, b_1, c_1 are linear forms, f_2 and h_2 is are homogeneous polynomials of degree two.

Proposition 7.1. The equality $f_2(z, t) = 0$ holds.

Let us prove Proposition 7.1 by reductio ad absurdum. Suppose that $f_2(z, t) \neq 0$.

Remark 7.2. By choosing suitable coordinates, we may assume that $f_2 = zt$ or $f_2 = z^2$.

We must use smoothness of the threefold X by analyzing the shape of q_4. We have

$$q_4 = f_4\big(z,t\big) + xu_3\big(z,t\big) + yv_3\big(z,t\big) + x^2a_2(x,y,z,t) + xyb_2(x,y,z,t) + y^2c_2(y,z,t),$$

where a_2, b_2, c_2 are homogeneous polynomials of degree two, u_3 and v_3 are homogeneous polynomials of degree three, and f_4 is a homogeneous polynomial of degree four.

Lemma 7.3. Suppose that $f_2(z, t) = zt$ and

$$f_4\big(z,t\big) = t^2g_2\big(z,t\big)$$

for some $g_2(z, t) \in \mathbb{C}[z, t]$. Then $v_3(z, 0) \neq 0$.

Proof. Suppose that $v_3(z, 0) = 0$. The surface T is given by the equation

$$w^2y^2+yzt+y^2c_1\big(x,y,z,t\big)+t^2g_2\big(z,t\big)+yv_3\big(z,t\big)+y^2c_2\big(x,y,z,t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[y,z,t,w]\Big) \cong \mathbb{P}^3$$

because T is cut out on X by the equation $x = 0$. Then T has non-isolated singularity along the line $x = y = t = 0$, which is impossible because X is smooth. $\square$

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

Corollary 7.4. Suppose that $f_2(z, t) = zt$ and

$$f_4\big(z,t\big) = z^2g_2\big(z,t\big)$$

for some $g_2(z, t) \in \mathbb{C}[z, t]$. Then $v_3(0, t) \neq 0$.

Lemma 7.5. Suppose that $f_2(z, t) = zt$. Then $f_4(0, t) = f_4(z, 0) = 0$.

Proof. We may assume that $f_4(z, 0) \neq 0$. Let $\mathcal{H}$ be the linear system on X that is cut out by

$$\lambda x + \mu y + \nu t = 0,$$

where $(\lambda : \mu : \nu) \in \mathbb{P}^2$. Then the base locus of $\mathcal{H}$ consists of the point P.

Let $\mathcal{R}$ be a proper transform of $\mathcal{H}$ on the threefold V. Then the base locus of $\mathcal{R}$ consists of a single point that is not contained in any of the curves $\bar{L}_1, \ldots, \bar{L}_r$.

The linear system $\mathcal{R}|_B$ has not base points, where B is a general surface in $\mathcal{B}$. But

$$R \cdot R \cdot B = 2n > 0,$$

where R is a general surface in $\mathcal{R}$. Then $\mathcal{R}|_B$ is not composed from a pencil, which implies that the curve $R \cdot B$ is irreducible and reduced by the Bertini theorem.

Let H and M be general surfaces in $\mathcal{H}$ and $\mathcal{M}$, respectively. Then $M \cdot H$ is irreducible and reduced. Thus, the linear system $\mathcal{M}|_H$ is a pencil.

The surface H contains no lines passing through P, and H can be given by

$$w^3x+w^2y^2+w\Big(y^2l_1(x,y,z)+xl_2(x,y,z)\Big)+l_4(x,y,z)=0\subset \mathrm{Proj}\Big(\mathbb{C}[x,y,z,w]\Big)\cong\mathbb{P}^3,$$

where $l_i(x,y,z)$ is a homogeneous polynomials of degree i.

Arguing as in Example 1.4, we see that there is a pencil $\mathcal{Q}$ on the surface H such that

$$\mathcal{Q}\sim\mathcal{O}_{\mathbb{P}^3}(2)\Big|_H,$$

general curve in $\mathcal{Q}$ is irreducible, and $\mathrm{mult}_P(\mathcal{Q})=4$. Arguing as in the proof of Lemma 3.1, we see that $\mathcal{M}|_H=\mathcal{Q}$ by [2, Theorem 2.2]. Let M be a general surface in $\mathcal{M}$. Then

$$M\equiv -2K_X,$$

and $\mathrm{mult}_P(M)=4$. The surface M is cut out on X by an equation

$$\lambda x^2+x\Big(A_0+A_1(y,z,t)\Big)+B_2(y,z,t)+B_1(y,z,t)+B_0=0,$$

where A_i and B_i are homogeneous polynomials of degree i, and $\lambda\in\mathbb{C}$.

It follows from $\mathrm{mult}_P(M)=4$ that $B_1(y,z,t)=B_0=0$.

The coordinated (y,z,t) are also local coordinates on X near the point P. Then

$$x=-y^2-y\Big(zt+yp_1(y,z,t)\Big)+\text{higher order terms},$$

which is a Taylor power series for $x=x(y,z,t)$, where $p_1(y,z,t)$ is a linear form.

The surface M is locally given by the analytic equation

$$\lambda y^4+\Big(-y^2-yzt-y^2p_1(y,z,t)\Big)\Big(A_0+A_1(y,z,t)\Big)+B_2(y,z,t)+\text{higher order terms}=0,$$

and $\mathrm{mult}_P(M)=4$. Hence, we see that $B_2(y,z,t)=A_0y^2$ and

$$A_1(y,z,t)y^2+A_0y\Big(zt+yp_1(y,z,t)\Big)=0,$$

which implies that $A_0=A_1(y,z,t)=B_2(y,z,t)=0$. Hence, we see that a general surface in the pencil $\mathcal{M}$ is cut out on X by the equation $x^2=0$, which is a absurd. $\square$

Arguing as in the proof of Lemma 7.5, we obtain the following corollary.

Corollary 7.6. Suppose that $f_2(z,t)=z^2$. Then $f_4(0,t)=0$.

Let $\mathcal{R}$ be the linear system on the threefold X that is cut out by cubics

$$xh_2(x,y,z,t)+\lambda\Big(w^2x+wy^2+q_3(x,y,z,t)\Big)=0,$$

where h_2 is a form of degree 2, and $\lambda\in\mathbb{C}$. Then $\mathcal{R}$ has no fixed components.

Let M and R be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M\cdot R=\sum_{i=1}^{r}m_iL_i+\Delta,$$

where $m_i\in\mathbb{N}$, and Δ is a cycle, whose support contains no lines among $L_1,\ldots,L_r$.

Lemma 7.7. The cycle Δ is not trivial.

Proof. Suppose that $\Delta=0$. Then $\mathcal{M}=\mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil. $\square$

We have $\mathrm{mult}_P(\Delta) \geqslant 8n - \sum_{i=1}^r m_i$, because $\mathrm{mult}_P(\mathcal{M}) = 2n$ and $\mathrm{mult}_P(\mathcal{R}) \geqslant 4$. Then

$$\deg\big(\Delta\big) = 12n - \sum_{i=1}^r m_i \geqslant 2\mathrm{mult}_P\big(\Delta\big) \geqslant 2\Big(8n - \sum_{i=1}^r m_i\Big)$$

by Lemma 4.5, because $\mathrm{Supp}(\Delta)$ does not contain any of the lines $L_1, \ldots, L_r$.

Corollary 7.8. The inequality $\sum_{i=1}^r m_i \geqslant 4n$ holds.

Let R_1 and R_2 be general surfaces in the linear system $\mathcal{R}$. Then

$$m_i \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)\mathrm{mult}_{L_i}(M) \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2$$

for every $1 \leqslant i \leqslant 4$ by Lemmas 2.1 and 3.3. Then

$$4n \leqslant \sum_{i=1}^r m_i \leqslant \sum_{i=1}^r \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2.$$

Corollary 7.9. The inequality $\sum_{i=1}^r \mathrm{mult}_{L_i}(R_1 \cdot R_2) \geqslant 8$ holds.

Now we suppose that R_1 is cut out on the quartic X by the equation

$$w^2x + wy^2 + q_3\big(x, y, z, t\big) = 0,$$

and R_2 is cut out by $xh_2\big(x, y, z, t\big) = 0$, where h_2 is sufficiently general. Then

$$\sum_{i=1}^r \mathrm{mult}_{L_i}\Big(R_1 \cdot T\Big) = \sum_{i=1}^r \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big) \geqslant 8,$$

where T is the hyperplane section of the hypersurface X that is cut out by $x = 0$. But

$$R_1 \cdot T = Z_1 + Z_2,$$

where Z_1 and Z_2 are cycles on X such that Z_1 is cut out by $x = y = 0$, and Z_2 is cut out by

$$x = wy + f_2\big(z, t\big) + yc_1\big(x, y, z, t\big) = 0.$$

Lemma 7.10. The equality $\sum_{i=1}^r \mathrm{mult}_{L_i}(Z_1) = 4$ holds.

Proof. The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = y = q_4\big(x, y, z, t\big) = 0,$$

which implies that $\sum_{i=1}^r \mathrm{mult}_{L_i}(Z_1) = 4$. □

Hence, we see that $\sum_{i=1}^r \mathrm{mult}_{L_i}(Z_2) \geqslant 4$. But Z_2 can be considered as a cycle

$$wy+f_2\big(z,t\big)+yc_1\big(y,z,t\big) = f_4\big(z,t\big)+yv_3\big(z,t\big)+y^2c_2(y,z,t) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[y,z,t,w]\Big) \cong \mathbb{P}^3,$$

and, putting $u = w + c_1(y, z, t)$, we see that Z_2 can be considered as a cycle

$$uy + f_2\big(z, t\big) = f_4\big(z, t\big) + yv_3\big(z, t\big) + y^2c_2(y, z, t) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[y, z, t, u]\Big) \cong \mathbb{P}^3,$$

and we can consider the set of lines $L_1, \ldots, L_r$ as the set in $\mathbb{P}^3$ given by $y = f_4(z, t) = 0$.

Lemma 7.11. The inequality $f_2(z, t) \neq zt$ holds.

Proof. Suppose that $f_2(z,t)=zt$. Then it follows from Lemma 7.5 that

$$f_4\big(z,t\big)=zt\big(\alpha_1 z+\beta_1 t\big)\big(\alpha_2 z+\beta_2 t\big)$$

for some $(\alpha_1:\beta_1)\in\mathbb{P}^1\ni(\alpha_2:\beta_2)$. Then Z_2 can be given by

$$uy+zt=yv_3\big(z,t\big)+y^2c_2\big(y,z,t\big)-uy\big(\alpha_1 z+\beta_1 t\big)\big(\alpha_2 z+\beta_2 t\big)=0\subset\mathrm{Proj}\Big(\mathbb{C}[y,z,t,u]\Big)\cong\mathbb{P}^3,$$

which implies $Z_2=Z_2^1+Z_2^2$, where Z_2^1 and Z_2^2 are cycles in $\mathbb{P}^3$ such that Z_2^1 is given by

$$y=uy+zt=0,$$

and Z_2^2 is given by $uy+zt=v_3(z,t)+yc_2(y,z,t)-u(\alpha_1 z+\beta_1 t)(\alpha_2 z+\beta_2 t)=0$.

We may assume that L_1 is given by $y=z=0$, and L_2 is given by $y=t=0$. Then

$$Z_2^1=L_1+L_2,$$

which implies that $\sum_{i=1}^r\mathrm{mult}_{L_i}(Z_2^2)\geqslant 2$.

Suppose that $r=4$. Then $\alpha_1\neq 0$, $\beta_1\neq 0$, $\alpha_2\neq 0$, $\beta_2\neq 0$. Hence, we see that

$$L_1\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_2,$$

because $v_3(z,t)+yc_2(y,z,t)-u(\alpha_1 z+\beta_1 t)(\alpha_2 z+\beta_2 t)$ does not vanish on L_1 and L_2. But

$$L_3\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_4,$$

because zt does not vanish on L_3 and L_4. Then $\sum_{i=1}^r\mathrm{mult}_{L_i}(Z_2^2)=0$, which is impossible.

Suppose that $r=3$. We may assume that $(\alpha_1,\beta_1)=(1,0)$, but $\alpha_2\neq 0\neq\beta_2$. Then

$$L_2\not\subseteq\mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t)+yc_2(y,z,t)-uz(\alpha_2 z+\beta_2 t)$ does not vanish on L_2. We have

$$f_4\big(z,t\big)=z^2t(\alpha_2 z+\beta_2 t),$$

which implies that $v_3(0,t)\neq 0$ by Corollary 7.4. Hence, wee see that

$$L_1\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_3,$$

because $v_3(z,t)+yc_2(y,z,t)-uz(\alpha_2 z+\beta_2 t)$ and zt do not vanish on L_1 and L_3, respectively, which implies that $\sum_{i=1}^r\mathrm{mult}_{L_i}(Z_2^2)=0$. The latter is a contradiction.

We see that $r=2$. We may assume that $(\alpha_1,\beta_1)=(1,0)$, and either $\alpha_2=0$ or $\beta_2=0$.

Suppose that $\alpha_2=0$. Then $f_4(z,t)=\beta_2 z^2t^2$. By Lemma 7.3 and Corollary 7.4, we get

$$v_3\big(0,t\big)\neq 0\neq v_3\big(z,0\big),$$

which implies that $v_3(z,t)+yc_2(y,z,t)-\beta_2 zt$ does not vanish on neither L_1 nor L_2. Then

$$L_1\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_2,$$

which implies that $\sum_{i=1}^r\mathrm{mult}_{L_i}\big(Z_2^2\big)=0$, which is a contradiction.

We see that $\alpha_2\neq 0$ and $\beta_2=0$. We have $f_4(z,t)=\alpha_2 z^3t$. Then

$$v_3\big(0,t\big)\neq 0$$

by Corollary 7.4. Then $L_1\not\subseteq\mathrm{Supp}(Z_2^2)$ because the polynomial

$$v_3(z,t)+yc_2(y,z,t)-\alpha_2 z^2$$

does not vanish on L_1.

The line L_2 is given by the equations $y=t=0$. But Z_2 is given by the equations

$$uy+zt=v_3\big(z,t\big)+yc_2\big(y,z,t\big)-\alpha_2 uz^2=0,$$

which implies that $L_2 \not\subseteq \mathrm{Supp}(Z_2^2)$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. $\square$

Therefore, we see that $f_2(z,t) = z^2$. It follows from Corollary 7.6 that

$$f_4\big(z,t\big) = zg_3\big(z,t\big)$$

for some $g_3(z,t) \in \mathbb{C}[z,t]$. We may assume that L_1 is given by $y = z = 0$.

Lemma 7.12. The equality $g_3(0,t) = 0$ holds.

Proof. Suppose that $g_3(0,t) \neq 0$. Then $\mathrm{Supp}(Z_2) = L_1$, because Z_2 is given by

$$uy + z^2 = zg_3\big(z,t\big) + yv_3\big(z,t\big) + y^2c_2(y,z,t) = 0,$$

and the lines $L_2, \ldots, L_r$ are given by the equations $y = g_3(z,t) = 0$.

The cycle $Z_2 + L_1$ is given by the equations

$$uy + z^2 = z^2g_3\big(z,t\big) + zyv_3\big(z,t\big) + zy^2c_2\big(y,z,t\big) = 0,$$

which implies that the cycle $Z_2 + L_1$ can be given by the equations

$$uy + z^2 = zyv_3\big(z,t\big) + zy^2c_2\big(y,z,t\big) - uyg_3\big(z,t\big) = 0.$$

We have $Z_2 + L_1 = C_1 + C_2$, where C_1 and C_2 are cycles in $\mathbb{P}^3$ such that C_1 is given by

$$y = uy + z^2 = 0,$$

and the cycle C_2 is given by the equations

$$uy + z^2 = zv_3\big(z,t\big) + zyc_2\big(y,z,t\big) - ug_3\big(z,t\big) = 0.$$

We have $C_1 = 2L_2$. But $L_1 \not\subseteq \mathrm{Supp}(C_2)$ because the polynomial

$$zv_3\big(z,t\big) + zyc_2\big(y,z,t\big) - ug_3\big(z,t\big)$$

does not vanish on L_1, because $g_3(0,t) \neq 0$. Then

$$Z_2 + L_1 = 2L_2,$$

which implies that $Z_2 = L_1$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2) = 1$, which is a contradiction. $\square$

Thus, we see that $r \leqslant 3$ and

$$f_4\big(z,t\big) = z^2\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big)$$

for some $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$. Then

$$v_3\big(0,t\big) \neq 0$$

by Corollary 7.4. But Z_2 can be given by the equations

$$uy+z^2 = yv_3\big(z,t\big)+y^2c_2\big(y,z,t\big)-uy\big(\alpha_1 z+\beta_1 t\big)\big(\alpha_2 z+\beta_2 t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[y,z,t,u]\Big) \cong \mathbb{P}^3,$$

which implies $Z_2 = Z_2^1 + Z_2^2$, where Z_2^1 and Z_2^2 are cycles on $\mathbb{P}^3$ such that Z_2^1 is given by

$$y = uy + z^2 = 0,$$

and the cycle Z_2^2 is given by the equations

$$uy + z^2 = v_3\big(z,t\big) + yc_2\big(y,z,t\big) - u\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big) = 0,$$

which implies that $Z_2^1 = 2L_1$. Thus, we see that $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) \geqslant 2$.

Lemma 7.13. The inequality $r \neq 3$ holds.

Proof. Suppose that $r = 3$. Then $\beta_1 \neq 0 \neq \beta_2$, which implies that

$$L_1 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on L_1. But

$$L_2 \not\subseteq \mathrm{Supp}\big(Z_2^2\big) \not\supseteq L_3,$$

because $\beta_1 \neq 0 \neq \beta_2$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. $\square$

Thus, we see that either $r = 1$ or $r = 2$.

Lemma 7.14. The inequality $r \neq 2$ holds.

Proof. Suppose that $r = 2$. We may assume that

- either $\beta_1 \neq 0 = \beta_2$,
- or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$.

Suppose that $\beta_2 = 0$. Then $f_4(z,t) = \alpha_2 z^3(\alpha_1 z + \beta_1 t)$ and

$$L_1 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t) + yc_2(y,z,t) - \alpha_2 uz(\alpha_1 z + \beta_2 t)$ does not vanish on L_1. But L_2 is given by

$$y = \alpha_1 z + \beta_1 t = 0,$$

which implies that z^2 does not vanish on L_2, because $\beta_1 \neq 0$. Then

$$L_2 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

which implies that $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Hence, we see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$. Then $L_1 \not\subseteq \mathrm{Supp}(Z_2^2)$, because

$$v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)^2$$

does not vanish on L_1. But $L_2 \not\subseteq \mathrm{Supp}(Z_2^2)$, because z^2 does not vanish on L_2. Then

$$\sum_{i=1}^{r} \mathrm{mult}_{L_i}\big(Z_2^2\big) = 0,$$

which is a contradiction. $\square$

We see that $f_4(z,t) = z^2$ and $f_4(z,t) = \mu z^4$ for some $0 \neq \mu \in \mathbb{C}$. Then Z_2^2 is given by

$$uy + z^2 = v_3\big(z,t\big) + yc_2\big(y,z,t\big) - \mu z^2 = 0,$$

where $v_3(0,t) \neq 0$ by Corollary 7.4. Thus, we see that $L_1 \not\subseteq \mathrm{Supp}(Z_2^2)$, because

$$v_3\big(z,t\big) + yc_2\big(y,z,t\big) - \mu z^2$$

does not vanish on L_1. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

The assertion of Proposition 7.1 is proved.

The assertion of Theorem 1.5 follows from Propositions 3.4, 5.1, 6.1, 7.1.

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